

F-Theory Vacua with \mathbb{Z}_3 Gauge Symmetry

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Abstract

Discrete gauge groups naturally arise in F-theory compactifications on genus-one fibered Calabi-Yau manifolds. Such geometries appear in families that are parameterized by the Tate-Shafarevich group of the genus-one fibration. While the F-theory compactification on any element of this family gives rise to the same physics, the corresponding M-theory compactifications on these geometries differ and are obtained by a fluxed circle reduction of the former. In this note, we focus on an element of order three in the Tate-Shafarevich group of the general cubic. We discuss how the different M-theory vacua and the associated discrete gauge groups can be obtained by Higgsing of a pair of five-dimensional $U(1)$ symmetries. The Higgs fields arise from vanishing cycles in I_2 -fibers that appear at certain codimension two loci in the base. We explicitly identify all three curves that give rise to the corresponding Higgs fields. In this analysis the investigation of different resolved phases of the underlying geometry plays a crucial rôle.

1 Introduction

Discrete symmetries play a key rôle for constructing extensions of the standard model of particle physics. In particular, discrete symmetries are used to forbid terms in the MSSM superpotential that would allow for fast proton decay or other processes which are highly suppressed in the standard model. Well known examples are provided by R-parity (\mathbb{Z}_2), baryon triality (\mathbb{Z}_3) and proton hexality (\mathbb{Z}_6) [1–3]. Conceptually, these discrete symmetries must be realized as discrete gauge symmetries arising, for example, as remnants of a broken $U(1)$ symmetries because global discrete symmetries cannot exist in a theory of quantum gravity [4].

The understanding of the geometrical origin of discrete gauge symmetries in F-theory compactifications is therefore of crucial interest both for conceptual as well as for phenomenological reasons. Here, discrete gauge symmetries arise from Calabi-Yau geometries which are only genus-one fibrations without section, in contrast to elliptic fibrations with sections. Recently, there has been progress in understanding the physics of such compactifications, starting with [6] and followed by [7–13]. A natural object which is attached to this kind of compactifications is given by the Tate-Shafarevich (TS) group of the genus-one fibration which is a discrete group that organizes inequivalent genus-one geometries which share the same associated Jacobian fibration. An F-theory compactification on a genus-one fibration does only depend on the Weierstrass equation, that is the Jacobian fibrations, and is, thus, insensitive to the chosen element of the TS-group. The TS-group determines

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the resulting discrete gauge symmetries of the F-theory compactification. In contrast, M-theory compactifications on different elements in the TS-group of the genus-one fibration are physically distinct. This is consistent with M-/F-theory duality due to the additional degree of freedom of the Wilson line of the discrete group in the circle compactification from F-theory to M-theory [5].

Genus-one fibrations do not possess a section, but only multi-sections. A multi-section defining an n -fold branched cover over the base of the genus-one fibration is referred to as a n -section. The moduli space of F-theory compactifications on genus-one fibrations with an n -section is connected by an extremal transition to that of compactifications on elliptic fibrations with n sections. Explicit models with up to three U(1) factors have been systematically constructed and analyzed for F-theory in [14–17]. This allows an analysis of the physics using the two dual pictures before and after the transition. This transition has been identified in the analyzed situations and for Calabi-Yau threefolds as a conifold transition [8, 10]. The vanishing cycles in this transition are rational curves that appear as components of I_2 -fibers which occur at certain codimension two loci in the base of the elliptic fibration. The resulting singular geometry is deformed by a three-sphere S^3 which glues several sections into a multi-section [7]. From the effective field theory point of view, in the blow-down appear a number of massless fields from M2-branes wrapping the vanishing cycle, that acquire a vacuum expectation value (VEV) corresponding to the deformation. In particular, the Higgsing of a $U(1)^n$ symmetry to \mathbb{Z}_n symmetry requires for the final step in the chain of Higgsings with a single remaining U(1) a massless scalar field of charge n . When compactifying on a circle to M-theory, there is a choice of Wilson line for this U(1). There are n different choices to turn on a Wilson line along the compactification circle yielding n different vacua of M-theory. Indeed, as this Wilson line also affects the mass formula for the Kaluza-Klein modes one can see that there will be n inequivalent Kaluza Klein towers that can arise in five dimensions. Notably for each choice of Wilson line, there is one Kaluza-Klein mode within each tower that becomes massless and serves as five-dimensional Higgs field which - once they acquire a vacuum expectation value - lead to n different five-dimensional vacua. These are to be compared with the n different M-theory compactifications on the respective elements of the TS group. Geometrically, one thus has to identify n different vanishing curves that correspond to these n Kaluza Klein modes of the Higgs field.

Most of the recent works have focused on the case of the gauge group \mathbb{Z}_2 which is the simplest case to consider as the TS-group consists in this case only of the genus one fibration (realized as a hypersurface in $\mathbb{P}[1, 1, 2]$) and its Jacobian. Accordingly, it is possible to identify the two corresponding Higgs fields as arising from the two rational components of the same I_2 -fiber [11, 12].

In this note we extend the above analysis beyond \mathbb{Z}_2 concentrating mainly on the case of \mathbb{Z}_3 which arises from the most general cubic admitting a trisection only, although we expect our findings will also have to play a key role for the understanding of TS-groups of higher order. In the case of \mathbb{Z}_3 , there are three different M-theory vacua. We pursue the strategy to geometrically identify three different curves that give rise to the three different 5D Higgs fields in M-theory that lead to these inequivalent vacua. The main obstacle in identifying these curves is the fact all three curves should be located in the reducible I_2 -fiber at the same codimension two locus in the two-dimensional base. However, an I_2 -fiber naively has only two rational curves, so that the presence of the wanted third curve is elusive. The key result of this note is to identify the third rational curve inside the I_2 -fiber.

We will use two different methods to demonstrate its existence. As a first evidence, we compute a non-zero Gromov-Witten invariant for the elliptic fibration in the class of the purported third curve, that, as expected, agrees with the number of the relevant I_2 -loci. As a stronger check, we consider two different phases of the resolution of the elliptic fibration before the Higgsing. The corresponding geometry is the dP_1 -elliptic fibration considered in [9]. In other words, we extend the above conifold picture by also taking flop transitions⁶ into account⁷. In one resolution phase, we are able to manifestly see two curves of the I_2 -fiber leading to two inequivalent vacua after the transition, while the third curve remains invisible. In the second resolution phase, we see again two curves in the I_2 -fiber, however, one of which was not present in the first phase and leading to

⁶It is worth mentioning that we have to consider a non-toric flop.

⁷A similar phenomenon has already been observed in [21] in the context of tensionless strings in six and five dimensions. In that case too, M-theory is able to see the different phases of the theory arising from flop transitions, while these coincide in the six-dimensional F-theory compactification.

the *third* vacuum, while one of the original two curves in the first resolution has become invisible. In addition, we argue that there are also in the case of TS-group \mathbb{Z}_3 just two different geometries, namely the most general cubic and its Jacobian. However, the former can be equipped with two different actions of the Jacobian which make them differ as elements of the Tate Shafarevich group, thereby yielding the claimed three different M-theory vacua.

This note consists of two sections. In Section 2 we review some aspects of the TS-group and its appearance in M- and F-theory. In particular, we review how the different M-theory vacua can be obtained from a circle compactification with discrete choices for the Wilson line from the unique F-theory compactification, both before and after Higgsing. Then, in Section 3, we focus on the explicit construction of the three different curves supporting the Higgses which lead to the three different M-theory vacua. For this analysis we discuss two different resolution phases of the dP_1 geometry.

2 The Tate-Shafarevich group in M- and F-theory

In this section we discuss some background material that is necessary for our geometrical analysis in section 3. We start by elaborating on some mathematical facts about the Tate-Shafarevich group. Then, we continue with a review and slight clarification of the field theory formalisms in 6D and 5D developed in [8, 10–12].

As already discussed in the introduction, discrete groups naturally arise in F-theory compactifications on genus-one fibered Calabi-Yau manifolds X that admit instead of a globally well-defined section just an n -section. To every such genus-one fibration one can associate its corresponding Jacobian fibration $J(X)$, whose fibers are the Jacobian $J(C)$ of the genus one curve C . The Jacobian $J(C)$ is given by the degree zero part of the Picard group $\text{Pic}(C)$ which has a distinguished point given by the trivial line bundle. Thus, $J(C)$ is an elliptic curve and $J(X)$ an elliptic fibration. In addition, the Jacobian $J(C)$ equips C with the structure of a homogeneous space over $J(C)$. This additional structure is crucial for the notion of the TS-group $\text{III}(J(X))$ associated to the Jacobian fibration which comprises all genus one fibrations that share the same Jacobian. While the latter statement describes the TS-group only as a set, the group structure is related to the Jacobian action on the different genus one fibrations $\text{III}(J(X))$ and is discussed in further detail in appendix A. Since the discriminant locus and τ function of every element X in $\text{III}(J(X))$ are identical to those of $J(X)$, their corresponding F-theory compactifications are identical. The TS-group $\text{III}(J(X))$ is manifest in the corresponding F-theory compactification as its discrete gauge group. In particular, we expect $\mathbb{Z}_n \subset \text{III}(J(X))$ for a genus-one fibration X with only an n -section and the discrete gauge group of F-theory to contain a \mathbb{Z}_n -factor.

A different point of view on a genus-one fibration admitting an n -section is to start with an elliptic fibration with n rational sections. Then, one performs a chain of complex structure deformations that merge the n sections into a single n -section. Technically, if we define the n sections as the n rational roots of a polynomial, we can understand this deformation as introducing additional terms to the polynomial so that its roots are no longer rational, but involve taking roots. In other words, the sections can only be defined over a certain field extension⁸. Upon considering a family of these polynomials parametrized by the base of the elliptic fibration, the n individual roots are not well-defined anymore due to branch cuts around which they are exchanged by monodromies. Only monodromy-invariant combinations are globally well-defined in the fibration, giving rise to the n -section which is an appropriate union of the original roots.

In contrast to the F-theory picture, one obtains different vacua by compactifying the dual M-theory on the corresponding elements of the Tate-Shafarevich group. These are related to the F-theory vacuum by an additional S^1 -compactification in which we in addition specify a certain flux choice along the compactification circle. This flux choice introduces the sought for additional degree of freedom, that is needed to reproduce the different lower-dimensional M-theory vacua starting from

⁸In general, given a field K with closure \bar{K} , a divisor D is defined over K , if $\sigma(D) = D$ for all elements σ in the absolute Galois group $G_{\bar{K}/K}$. Also note that, as we are dealing with a function field associated to the two-dimensional base, the notion of roots refers to taking suitable covers of the base.

a unique F-theory compactification before circle compactification, as we explain next.

The following, purely field theoretical analysis of the emergence of the different M-theory vacua has been proposed in [8, 10–12]. Here, we review the aspects essential to our discussion and elucidate some additional important insights. Although the following similarly applies to four-dimensional F-theory compactification, we focus here on F-theory in 6D with an M-theory dual in 5D for clarity of our discussion. The crucial point in the following is the fundamental difference between the Higgs effect in 6D and in 5D due to the presence of another U(1) gauge field in 5D, that is the Kaluza-Klein U(1).

The starting point is a six-dimensional U(1)_{6d} gauge theory that we want to break by a Higgs field Φ of charge q to \mathbb{Z}_q . As a next step, we compactify this theory on a circle S^1 . The Higgs field enjoys accordingly an expansion of the form

$$\Phi(x, y) = \sum_{n \in \mathbb{Z}} \phi_n(x) e^{2\pi i n y}. \quad (2.1)$$

In addition, one can turn on a flux (Wilson line) $\xi = \int_{S^1} A_{6d}$ along the S^1 . The full mass formula for the n th KK-mode then reads

$$m_n^q = |q\xi + n|, \quad (2.2)$$

where q and n refer to the charges under U(1)_{6d} and U(1)_{KK}, the Kaluza-Klein (KK) U(1) gauge field, respectively.

It is crucial to note that these gauge groups and their corresponding charges are only well-defined up to unimodular transformations that act on the charge vector (q, n) and the two U(1) fields as

$$(q \quad n) \mapsto (q \quad n) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \begin{pmatrix} \text{U}(1)_{6d} \\ \text{U}(1)_{KK} \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \text{U}(1)_{6d} \\ \text{U}(1)_{KK} \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \quad (2.3)$$

These transformations allow us to identify different 5D KK-towers that arise from different choices for the flux ξ . Indeed, if one re-defines the five-dimensional U(1) fields according to

$$\begin{pmatrix} \text{U}(1)_{6d} \\ \text{U}(1)_{KK} \end{pmatrix} \mapsto \begin{pmatrix} 1 & -r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{U}(1)_{6d} \\ \text{U}(1)_{KK} \end{pmatrix} \quad (2.4)$$

while shifting $\xi \mapsto \xi + r$ at the same time, the mass formula (2.2) is invariant. Thus, one obtains again the same KK-tower.

Varying the circle-flux ξ , we see that we have phase transitions at $\xi = \frac{k}{q}$ for $k \in \mathbb{Z}$ since the KK-tower gets shifted, according to (2.2). For a given flux ξ , the mode ϕ_n triggering the Higgsing in 5D, which has to be chosen so that its mass in (2.2) vanishes, is changed. Each different choice for the 5D Higgs yields a *different* theory after Higgsing. Thus, we see that there are q inequivalent Higgses (for each different value of ξ) and corresponding five-dimensional vacua after Higgsing. These q different vacua precisely correspond to the different M-theory compactifications obtained from the different elements of the TS-group.

As a concrete example, let us focus on the case of interest, which is 6D F-theory compactification with one U(1)_{6d} with a Higgs Φ of charge $q = 3$. This is realized by F-theory compactified on dP_1 -elliptic fibrations [9] and the Higgsed theory has a \mathbb{Z}_3 discrete group. In 5D, one obtains the following gauge groups depending on the KK-charge of the 5D Higgs:

$$\begin{cases} U(1) \times \mathbb{Z}_3, & \text{if } n = 0 \bmod 3 \\ U(1), & \text{if } n \neq 0 \bmod 3 \end{cases}. \quad (2.5)$$

By the above discussion, we expect $q = 3$ M-theory vacua with massless Higgses fixed by (2.2) for $\xi = 0, \frac{1}{3}, \frac{2}{3}$, respectively. By (2.5) we see that only one of which has discrete gauge group \mathbb{Z}_3 and two of which have no discrete gauge group. Geometrically, this is expected as the TS-group of the cubic, which is the geometry obtained by deforming the dP_1 -elliptic fibration in the Higgsing [9], contains \mathbb{Z}_3 , see Appendix A. The geometric identification of these three Higgs field in terms of holomorphic curves wrapped by M2-branes is the subject of Section 3.

Before turning to this discussion, we would like to briefly mention a dual reformulations of the above picture in terms of a Stückelberg mechanism [8, 10]. We identify an axion as the phase of the six-dimensional Higgs field $\Phi = he^{ic}$. In the circle compactification from 6D to 5D, the axion c can acquire a vacuum expectation value

$$n = \int_{S^1} \langle dc \rangle \quad (2.6)$$

along the circle. Clearly, the VEV (2.6) signals a non-trivial profile of c along the S^1 , so that the flux (2.6) agrees with the KK mode number n of c . In this flux background, the surviving massless $U(1)$ is a linear combination of the two 5D $U(1)$ fields $U(1)_{6d}$ and $U(1)_{KK}$. Geometrically, a non-trivial vacuum expectation value of the axion is exactly expected to arise in compactifications involving multi-sections and takes into account the generalized T-duality transformation rules that govern the physics in the presence of off-diagonal terms in the metric [8].

This picture is dual to the above discussion with a corresponding Wilson line ξ that has to be chosen so that the massless Higgs by (2.2) yields precisely the same linear combination of $U(1)_{6d}$ and $U(1)_{KK}$ of the remaining massless $U(1)$ determined by the Stückelberg mechanism of the 5D axion c .

3 Identifying 5D Higgs Fields for the \mathbb{Z}_3 Tate-Shafarevich group

In this section we construct explicitly the different Higgs fields that lead to the three inequivalent five-dimensional M-theory vacua corresponding to the three different elements of the Tate-Shafarevich group of the most general cubic. These Higgs fields are obtained from M2-branes wrapping three different holomorphic curves in I_2 -fibers of the elliptic fibration with fiber in dP_1 , that we identify explicitly. F-theory compactifications both on Calabi-Yau fibrations with the most general cubic and the elliptic curve in dP_1 have been studied recently in [9] to which we refer for further details.

A genus-one fibration by the most general cubic in \mathbb{P}^2 is defined by the hypersurface

$$p_{\text{cub}} = s_1 u^3 + s_2 u^2 v + s_3 u v^2 + s_4 v^3 + s_5 u^2 w + s_6 u v w + s_7 v^2 w + s_8 u w^2 + s_9 v w^2 + s_{10} w^3 = 0. \quad (3.1)$$

Here the s_i are polynomials of appropriate degree of the coordinates on B or, more generally, sections of appropriate line bundles on B . By making specific choices for these line bundles and taking general sections, we obtain a genus one fibered Calabi-Yau threefold $X_{p_{\text{cub}}}$. Clearly, for generic s_i this genus one-fibration admits only a tri-section only.

Our strategy is to start in a six-dimensional F-theory compactification with a $U(1)$ gauge group that we want to break down to \mathbb{Z}_3 by a Higgsing. Such a geometry is provided by the dP_1 -elliptic fibration which has been considered in great detail in [9]. It is related to the geometry of the cubic by the tuning and resolution of the form

$$\text{cubic} \xrightarrow{s_{10} \rightarrow 0} \text{singular } dP_1 \xrightarrow{\text{blowing up } e_1} \text{resolved } dP_1, \quad (3.2)$$

which is geometrically a conifold transition. Thus, the singular dP_1 model is given (3.1) for $s_{10} = 0$,

$$p_{dP_1}^s = s_1 u^3 + s_2 u^2 v + s_3 u v^2 + s_4 v^3 + s_5 u^2 w + s_6 u v w + s_7 v^2 w + s_8 u w^2 + s_9 v w^2 = 0. \quad (3.3)$$

One observes that it has conifold singularities at co-dimension two in the base at $s_8 = s_9 = 0 = u = v$. In the following two subsections, we argue that this singularity admits two different resolutions related by a flop. These resolutions lead to the same F-theory vacuum, but the corresponding M-theory vacua are different as we demonstrate by analysing the charges of the corresponding I_2 -fibers in the respective resolutions.

The crucial point is that in the first resolution two holomorphic curves are manifest that correspond to two of the three wanted 5D Higgs fields, whereas the curve supporting the third Higgs is obscured. In contrast, in the second resolution, again two curves are manifest that, however, correspond to the *third* wanted 5D Higgs field, while one of the curves manifest in the first resolution is obscured now. Thus, we see that by using different phases of the resolution of the singularity at

$s_8 = s_9 = 0$ in the dP_1 -elliptic fibration, we can indeed manifestly see three different holomorphic curves to be used as the 5D Higgses yielding the three different M-theory vacua from compactification on the three elements of the TS-group of the cubic (3.1).

3.1 Resolving by the toric blow-up

The obvious resolution of the singularities in elliptic fibration (3.3) is obtained by performing the toric blow-up in the ambient space of the elliptic fiber defined by

$$u \longrightarrow e_1 u, \quad v \longrightarrow e_1 v, \quad (3.4)$$

where the coordinate e_1 vanishes at the exceptional divisor. This is precisely toric dP_1 , which yields a resolution of the singular fibration with the cubic (3.3). The corresponding smooth elliptic fibration has been extensively studied in [9] and we just summarize the most important results here. First of all the toric data of dP_1 is given by

$$\left(\begin{array}{cc|cc|c|c} 1 & 0 & 1 & 0 & w & U \\ -1 & 1 & 0 & 1 & u & U - S \\ -1 & 0 & 1 & -1 & e_1 & S \\ 0 & -1 & 0 & 1 & v & U - S \end{array} \right). \quad (3.5)$$

Here the first two columns display the points of the toric diagram, written as row vectors, followed by the generators of the Mori cone in the third and fourth column, while the last two columns refer to the coordinates and the corresponding divisor classes being assigned to the rays of the diagram, respectively. The fan of the toric variety dP_1 is obtained by a fine star-triangulation of the polytope defined by (3.5), from which one immediately obtains the Stanley Reisner ideal as well as the intersection numbers

$$SR = \{uv, e_1 w\}, \quad U^2 = 1, \quad S^2 = -1, \quad U \cdot S = 0. \quad (3.6)$$

The resolved curve takes the form

$$p_{dP_1}^r = s_1 u^3 e_1^2 + s_2 u^2 v e_1^2 + s_3 u v^2 e_1^2 + s_4 v^3 e_1^2 + s_5 u^2 w e_1 + s_6 u v w e_1 + s_7 v^2 w e_1 + s_8 u w^2 + s_9 v w^2 = 0. \quad (3.7)$$

Over the locus defined by $s_8 = s_9 = 0$, the fiber degenerates as

$$\underbrace{e_1}_{c_1} \underbrace{(s_1 u^3 e_1 + s_2 u^2 v e_1 + s_3 u v^2 e_1 + s_4 v^3 e_1 + s_5 u^2 w + s_6 u v w + s_7 v^2 w)}_{c_2} = 0, \quad (3.8)$$

which is a smooth I_2 -fiber. Here c_1 and c_2 denote the two \mathbb{P}^1 components of the resulting I_2 -fiber which have classes S and $3U - 2S$.

Next, we investigate the sections, starting from the singular model before the blow-up (3.4). It has two sections which take the form

$$S_0 = [0 : 0 : 1], \quad S_1 = \left[-s_9 : s_8 : \frac{s_4 s_8^3 - s_3 s_8^2 s_9 + s_2 s_8 s_9^2 - s_1 s_9^3}{-s_7 s_8^2 + s_6 s_8 s_9 - s_5 s_9^2} \right] \quad (3.9)$$

in terms of the projective coordinates $[u : v : w]$ on \mathbb{P}^2 . While the first section is toric and taken to define the zero-point O on the elliptic curve, the second section is found by constructing the tangent line

$$t_O : s_8 u + s_9 v = 0 \quad (3.10)$$

to the origin O which has to intersect the cubic in precisely one third point, provided $s_8 \neq 0$ or $s_9 \neq 0$. This third intersection point is rational and defines the section S_1 . In contrast, at the singular locus $s_8 = s_9 = 0$, the curve becomes singular precisely at O which implies that any line through O is tangent and therefore S_1 degenerates to the whole singular curve. The two curves c_1 and c_2 support the Higgs field in question.

| | c_1 | c_2 |
|-----------------------------|-------|-------|
| \tilde{S}_0 | -1 | 2 |
| \tilde{S}_1 | 2 | -1 |
| $\tilde{S}_1 - \tilde{S}_0$ | 3 | -3 |

Table 1: Intersection numbers of the sections \tilde{S}_0, \tilde{S}_1 with the curves in the I_2 -fiber at $s_8 = s_9 = 0$ and corresponding $U(1)_{6d}$ charges in the toric resolution of the dP₁-model.

Passing to the resolved geometry, the two sections (3.9) lift to

$$\tilde{S}_0 = [s_9 : -s_8 : 1 : 0], \quad \tilde{S}_1 = [-s_9 : s_8 : s_4 s_8^3 - s_3 s_8^2 s_9 + s_2 s_8 s_9^2 - s_1 s_9^3 : -s_7 s_8^2 + s_6 s_8 s_9 - s_5 s_9^2]. \quad (3.11)$$

Note that these expressions are only valid away from the locus $s_8 = s_9 = 0$. In fact, \tilde{S}_0 as well as \tilde{S}_1 are rational sections that wrap the components c_1 and c_2 completely, respectively. We note that the classes of the divisors associated to the sections are given by

$$[\tilde{S}_0] \cong S, \quad [\tilde{S}_1] \cong U - 2S, \quad (3.12)$$

where, by abuse of notation, we denoted the divisors S and U , defined in (3.5), on the ambient space by the same symbols as their intersections with the Calabi-Yau hypersurface (3.7). While the class of the zero section is obvious, the class of the second section is obtained by noting that the tangent line defines a tri-section. Subtracting from its class two times the class of the zero section⁹, one is left with $U - 2S$.

Knowing the classes of the sections as well as those of the fiber components c_1, c_2 , one can evaluate the corresponding charges w.r.t. the $U(1)_{6d}$ and the KK $U(1)$ -field $U(1)_{KK}$. Recalling that the charge under $U(1)_{6d}$ is computed from the intersection of the Shioda map for \tilde{S}_0 , that is $\tilde{S}_1 - \tilde{S}_0$, with the respective curve c_i while the KK-charge is given by the intersection number with the zero section \tilde{S}_0 , we obtain the charges summarized in Table 1. One observes that the shrinking of c_1 and c_2 gives massless states with charges $(-3, -1)$ and $(3, 2)$, respectively. According to (2.2), these states are massless for the associated (inverse) choices of flux $\xi_1 = -\frac{1}{3} \bmod 1$ and $\xi_2 = -\frac{2}{3} \bmod 1$. By (2.5) the remaining gauge group in 5D after Higgsing is $U(1)$.

Geometrically, we have to shrink the respective curves c_1 and c_2 and deform the arising singularity, cf. the inverse process of (3.2). While the shrinking of the first component c_1 is performed by taking the blow-down $e_1 \rightarrow 1$, the shrinking of the second component c_2 is more elaborate. Instead of describing this directly, we choose the following short-cut. Observe that the charges in Table 1 are symmetric in the curves c_1 and c_2 as well as in \tilde{S}_0 and \tilde{S}_1 . Therefore, we can also change the roles of \tilde{S}_0 and \tilde{S}_1 , i.e. we take \tilde{S}_1 to be the zero section¹⁰. Then, we obtain the second vacuum again by blowing down $e_1 \rightarrow 1$. Geometrically, this gives the same cubic after Higgsing which corresponds to switching on a non-zero value for s_{10} . However, this is precisely what we expect. The two non-trivial elements of the Tate Shafarevich group are given by the same cubic that only differs through inverse actions of the Jacobians, cf. Appendix A.

It remains to show how to obtain the massless mode of the Higgs that yields the third vacuum which must correspond to the Jacobian. The relevant Higgs field has to have charge $(3, 0)$ with $\xi = 0$, so that the gauge group is \mathbb{Z}_3 , according to (2.5). However, the associated curve is not manifest in the Calabi-Yau manifold (3.7). Nevertheless, one can argue for its existence by studying the enumerative geometry of the dP₁-fibration, which can be accomplished evaluating the non-perturbative part of the prepotential of the topological string,

$$F_{\text{non-pert}}^0 = \sum_{\beta} N_{\beta}^0 Q^{\beta}, \quad (3.13)$$

⁹Recall that the tangent line was defined in the singular geometry, where the class of u reads U instead of $U - S$.

¹⁰As shown in [22], such a change of the choice of zero section is always possible in an anomaly-free theory, which is the case for the theory at hand, cf. [9].

analogously to the analysis performed in [18]. Here, N_β^0 denote the (genus zero) Gromov-Witten invariants of the classes β .

To this end, we construct explicitly all Calabi-Yau manifolds (3.7) with base $B = \mathbb{P}^2$ following the algorithm in Appendix G of [23] and compute all Gromov-Witten invariants up to a suitable degree. We find that in all these cases the invariant $N_{[c_1+T^2]} = [s_8] \cdot [s_9]$, i.e. agrees precisely with the number of I_2 -fibers at $s_8 = s_9 = 0$. This is a strong indication that there exists a third holomorphic curve of genus zero inside the I_2 -fiber (3.8) at $s_8 = s_9 = 0$, that is harder to manifestly see in geometry. In fact, using the classes (3.12) we check that this curve has precisely the right charges, $(q, n) = (3, 0)$, that are required for obtaining the third geometry according to (2.5).

Next we consider a different resolution which makes the curves in the classe $[c_1 + T^2]$ directly manifest in the geometry.

3.2 Resolving by a complete intersection resolution

Let us begin by noting that in the toric blow-up (3.4), the zero section is ill-behaved, i.e. wraps a fiber component, precisely at the I_2 -locus of interest at $s_8 = s_9 = 0$. Therefore, the two manifest curves c_1 and c_2 in (3.8) have to have a non-trivial KK-charge by construction. One way to manifestly see the third curve at $s_8 = s_9 = 0$ with KK-charge zero is to consider a different phase of the resolution of the dP_1 -elliptic fibration (3.3) where the zero section remains *holomorphic*. In the following, we construct this resolution explicitly and show the existence of the desired curve with KK-charge zero. We note that such a resolution has been considered in a different context in [19, 24].

To get started, we rewrite the singular dP_1 geometry (3.3) in the form of a determinantal variety,

$$p_{dP_1}^s = uP_1 - vP_2. \quad (3.14)$$

Here we have defined the two summands as

$$P_1 = s_1u^2 + s_2uv + s_3v^2 + s_5uw + s_6vw + s_8w^2, \quad P_2 = -(s_4v^2 + s_7vw + s_9w^2). \quad (3.15)$$

For a variety of the form (3.14), we can perform a small resolution. This means we introduce the coordinates λ_1, λ_2 parameterizing a \mathbb{P}^1 and impose

$$u\lambda_1 = \lambda_2 P_2. \quad (3.16)$$

The proper transform of the dP_1 geometry (3.3) is accordingly given by

$$v\lambda_1 = \lambda_2 P_1. \quad (3.17)$$

In contrast to the toric blow-up in (3.4) that does not change the dimension of the ambient space, the two equations (3.16) and (3.17) define the resolved fiber as a complete intersection in a three-dimensional ambient space that can be torically characterized as

$$\left(\begin{array}{ccc|cc|c|c} -1 & -1 & -1 & 0 & 1 & u & L \\ -1 & 0 & 0 & 1 & -1 & \lambda_2 & H \\ 0 & 0 & 1 & 0 & 1 & v & L \\ 0 & 1 & 0 & 0 & 1 & w & L \\ 1 & 0 & 0 & 1 & 0 & \lambda_1 & H + L \end{array} \right). \quad (3.18)$$

Again, the first three columns specify the points of the toric diagram, written as row vectors, followed by the two generators of the Mori cone. The last two columns display the coordinates and the divisor classes which are associated to the rays, respectively. From the toric data one obtains immediately the Stanley Reisner ideal as well as the triple intersection numbers as

$$SR = \{\lambda_1 \lambda_2, uvw\}, \quad L^3 = 0, \quad L^2 \cdot H = 1, \quad L \cdot H^2 = -1, \quad H^3 = 1. \quad (3.19)$$

The evaluation of the irreducible components of the I_2 -fiber is slightly more involved in this case and is best performed using a primary decomposition. At the locus $s_8 = s_9 = 0$ we find that the two

| | c_+ | c_- |
|---------------|-------|-------|
| S'_0 | 1 | 0 |
| S'_1 | -2 | 3 |
| $S'_1 - S'_0$ | -3 | 3 |

Table 2: Intersection numbers of the sections S'_0, S'_1 with the curves in the I_2 -fiber at $s_8 = s_9 = 0$ and corresponding $U(1)_{6d}$ charges in the complete intersection resolution of the dP_1 -model.

rational components of the I_2 fiber are given by

$$\begin{aligned} c_+ &= V(u, v), \\ c_- &= V(s_4 v^2 \lambda_2 + s_7 v w \lambda_2 + u \lambda_1, s_1 u^2 \lambda_2 + s_2 u v \lambda_2 + s_3 v^2 \lambda_2 + s_5 u w \lambda_2 + s_6 v w \lambda_2 - v \lambda_1, \\ &\quad s_1 u^3 + s_2 u^2 v + s_3 u v^2 + s_4 v^3 + s_5 u^2 w + s_6 u v w + s_7 v^2 w, s_1 s_4 u v \lambda_2^2 + s_1 s_7 u w \lambda_2^2 \\ &\quad + s_4 s_5 v w \lambda_2^2 + s_5 s_7 w^2 \lambda_2^2 - s_2 u \lambda_1 \lambda_2 - s_3 v \lambda_1 \lambda_2 - s_6 w \lambda_1 \lambda_2 + \lambda_1^2). \end{aligned} \quad (3.20)$$

Here, $V(I)$ denotes the zero set of the ideal I and (p_1, \dots, p_k) denotes the ideal generated by the polynomials p_1, \dots, p_k . Applying again prime ideal techniques, one determines their respective homology classes of the curves c_+ and c_- as

$$[c_+] = L^2, \quad [c_-] = (2L + H)(2L + H) - L^2. \quad (3.21)$$

Next, we turn to the analysis of the sections (3.9). Outside $s_8 = s_9 = 0$ the simultaneous vanishing of u and v implies that $\lambda_2 = 0$ as well. Thus, we identify the coordinates $[u : v : w : \lambda_1 : \lambda_2]$ of the zero section and its homology class as

$$S'_0 = [0 : 0 : 1 : 1 : 0], \quad [S'_0] = H. \quad (3.22)$$

In contrast to the toric blow-up, S'_0 is also well-defined at the singular locus and defines, therefore, a holomorphic zero section. In order to find the second section, we note that away from the singular locus there is a unique solution for λ_1 and λ_2 if one plugs the coordinates of the section S_1 given in (3.9) into (3.16) and (3.17). Its coordinates are given by

$$\begin{aligned} S'_1 = [s_9 : -s_8 : &\frac{-s_4 s_8^3 + s_9(s_3 s_8^2 + s_9(-s_2 s_8 + s_1 s_9))}{s_7 s_8^2 + s_9(-s_6 s_8 + s_5 s_9)} : (-s_4 s_6 s_7 s_8^5 + s_3 s_7^2 s_8^5 + s_4^2 s_8^6 + s_4 s_6^2 s_8^4 s_9 \\ &+ s_4 s_5 s_7 s_8^4 s_9 - s_3 s_6 s_7 s_8^4 s_9 - s_2 s_7^2 s_8^4 s_9 - 2s_3 s_4 s_8^5 s_9 - 2s_4 s_5 s_6 s_8^3 s_9^2 + s_3 s_5 s_7 s_8^3 s_9^2 + s_2 s_6 s_7 s_8^3 s_9^2 \\ &+ s_1 s_7^2 s_8^3 s_9^2 + s_3^2 s_8^4 s_9^2 + 2s_2 s_4 s_8^4 s_9^2 + s_4 s_5^2 s_8^2 s_9^3 - s_2 s_5 s_7 s_8^2 s_9^3 - s_1 s_6 s_7 s_8^2 s_9^3 - 2s_2 s_3 s_8^3 s_9^3 + s_1^2 s_9^6 \\ &- 2s_1 s_4 s_8^3 s_9^3 + s_1 s_5 s_7 s_8 s_9^4 + s_2^2 s_8^2 s_9^4 + 2s_1 s_3 s_8^2 s_9^4 - 2s_1 s_2 s_8 s_9^5) : (s_7 s_8^2 + s_9(s_5 s_9 - s_6 s_8))^2]. \end{aligned} \quad (3.23)$$

A slightly alternative way to obtain this section is to compute the intersection of the hyperplane given by

$$s_8 u + s_9 v = 0 \quad (3.24)$$

with the complete intersection manifold specified by (3.16) and (3.17). Analogous to the case of the toric blow-up, its class is computed as

$$[S'_1] = [u] - 2[\tilde{S}_0] = L - 2H. \quad (3.25)$$

Finally, we compute the one finds the charges to the two rational curves c_+, c_- in the I_2 -fiber at $s_8 = s_9 = 0$ in this resolution. Here, we use their homology classes in (3.21) and the homology classes of the zero and rational sections in (3.22) and (3.25) to obtain the charges summarized in Table 2. In particular, we observe that one obtains a mode with KK-charge zero by shrinking the component c_- . It supports precisely the third Higgs necessary to obtain the third 5D M-theory vacuum with $U(1) \times \mathbb{Z}_3$ gauge group in (2.5). We emphasize that the curve c_- has precisely the same charges as the curve in the class $[c_1 + T^2]$ for whose existence we argued in the last subsection by computing the corresponding Gromov-Witten invariants. In analogy to the analysis performed in [11, 12], it is expected that the corresponding geometry after the deformation is given by the Jacobian of the most general cubic (3.1).

4 Conclusion and further directions

In this note we have considered the physics of F-theory compactifications with \mathbb{Z}_3 discrete gauge group using M-/F-theory duality. From a mathematical perspective this discrete gauge group arises from compactification on a genus-one fibered Calabi-Yau manifold with Tate-Shafarevich group \mathbb{Z}_3 . While the effective physics of the F-theory compactification is the same, regardless of which element of $\text{III}(J(X))$ is chosen, the compactification of M-theory to 5D on these three Calabi-Yau manifolds gives rise to three physically different vacua. We understand the emergence of these three different M-theory vacua by starting with an F-theory compactification with a $U(1)$ and matter with charge three obtained by compactification on the dP_1 -elliptic fibration considered in [9]. In 6D, Higgsing the $U(1)$ by the charged matter yields a \mathbb{Z}_3 discrete gauge group. In 5D, the three different vacua arise by three different Higgs effects. When performing a fluxed circle compactification of the F-theory *with* the $U(1)$ gauge symmetry, there are three different flux choices ($U(1)$ Wilson lines) that yield three different possible massless Higgs fields in 5D. The KK-labels of these Higgs fields are $n = 0, 1, 2$, respectively, that agree with their charge with respect to the 5D KK $U(1)$ field. The three different vacua of M-theory are then realized by these three different Higgsings: for the two Higgsing by the fields with $n = 1, 2$, we obtain a 5D theory with a remaining $U(1)$ gauge group, while we obtain a $U(1) \times \mathbb{Z}_3$ gauge group for the third Higgsing by the field with $n = 0$. The crucial advancement of our analysis is the explicit construction of the *three* different vanishing rational curves in a single I_2 -fiber of the dP_1 -elliptic fibration that yield the desired Higgs fields from wrapping M2-brane states.

We collect first evidence for the existence of three rational curves in a single I_2 -fiber in the dP_1 -elliptic fibration, by computation of the Gromov-Witten invariants in the three classes in which these curves are expected to be. In order to identify these three curves explicitly and the corresponding three different vacua, it was crucial to consider two different phases of the resolution of the dP_1 -elliptic fibration. The first one of these is the standard toric blow-up in dP_1 . Due to the fact that in this resolution the zero section is rational and wraps a fiber component precisely at the codimension two locus in the base, where the I_2 -fiber is located, only the curves supporting the Higgs fields with KK-charges $n = 1, 2$ are manifest. In contrast, the resolution in the second phase was performed using a small resolution, resulting in an elliptic fibration with elliptic fiber defined as a complete intersection in a three-dimensional ambient space. The key point is that in this fibration the zero section is holomorphic. Thus, the zero section is well-behaved everywhere on the base, in particular at the aforementioned I_2 -locus. Consequently, there exists a curve at the I_2 -locus with KK-charge $n = 0$, yielding the third Higgs, that was invisible in the first phase described by the toric blow-up. The geometry that arises by performing the conifold transition of this curve in the I_2 -locus is expected to be the Jacobian of the genus-one fibration, i.e. the trivial element in the TS-group. The other two geometries that arise from performing the conifold transition for the two curves with KK-charge $n = 1, 2$ in the toric blow-up, respectively, give both rise to the most general cubic corresponding to two M-theory vacua with no discrete gauge symmetry. However, their KK-towers are inequivalent. Mathematically, the only way to distinguish these two geometries is given by an additional piece of data given by the Jacobian action, which is in fact inverse for these two geometries.

While the analysis of the three different vacua has been successfully completed which was the main aim of this note, there are a list of interesting questions that demand further investigation and will be addressed in future works [25]:

1. What is the interpretation of the charge content in terms of intersection theory? It seems that is hard to achieve the charge content displayed in table 2 using the methods presented in [20] for rational curves with self-intersection -1 .
2. What is the physical interpretation of the Jacobian action on the genus-one geometries? This action is crucial to differentiate the two elements of the Tate Shafarevich group which are geometrically identical.
3. How does the group structure carried by $\text{III}(J(X))$ enter the physical discussion? It is tempting to conjecture that it is identical to the addition of the discrete Wilson line that is put on the circle used to compactify from F- to M-theory.

4. As the third M-theory vacuum corresponding to the mode with charges $(3,0)$ exhibits a \mathbb{Z}_3 -symmetry already in five dimensions, it is expected that the corresponding geometry has non-trivial torsional contributions to its homology lattice by analogy with the discussion in [12] on the \mathbb{Z}_2 case. It would be interesting to investigate the rôle of torsion in this example as well.

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A The Tate-Shafarevich group

Let $f_0 : X_0 \longrightarrow B$ be an elliptically fibred Calabi-Yau fibered over a base B . We define the Tate-Shafarevich group as a set as

$$\text{III}(X_0) = \{ \text{equivalence classes of pairs } (f, i), \text{ where } f : X \rightarrow B \text{ is a genus one fibration and } i : X_0 \rightarrow J(X/B) \text{ is an isomorphism.} \} \quad (\text{A.1})$$

Here we have denoted by $J(X/B)$ the relative Jacobian of X over B . That means that any element X of $\text{III}(X_0)$ is a homogeneous space over X_0 , i.e. we have a map

$$a_i : X_0 \times X \longrightarrow X \quad (\text{A.2})$$

such that the following properties hold

$$a_i(0, x_i) = x_i \quad \forall x_i \in X, \quad (\text{A.3})$$

$$a_i(x_0, a_i(\tilde{x}_0, x_i)) = a_i(x_0 + \tilde{x}_0, x_i), \quad (\text{A.4})$$

$$\forall x_1, \tilde{x}_1 \in X \exists! x_0 \in X_0 \text{ so that } a_i(x_0, x_1) = \tilde{x}_1. \quad (\text{A.5})$$

Therefore, by an element $X_i \in \text{III}(X_0)$ we mean the triple

$$\mathbf{X}_i = (X_i, f_i, a_i). \quad (\text{A.6})$$

We define the addition of two elements $\mathbf{X}_i, \mathbf{X}_j \in \text{III}(X_0)$ as follows. As a manifold, one has

$$X_{i+j} = (X_i \times_B X_j) / \sim = \{ (x_i, x_j) | f_i(x_i) = f_j(x_j) \} / \sim. \quad (\text{A.7})$$

Here, the equivalence relation on two points $(x_i, x_j), (y_i, y_j) \in X_i \times_B X_j$ is defined by

$$(x_i, x_j) \sim (y_i, y_j) \Leftrightarrow \exists x_0 \in X_0 \text{ s.t. } (a_i(x_0, x_i), a_j^{-1}(x_0, x_j)) = (y_i, y_j), \quad (\text{A.8})$$

where we have defined $a_i^{-1}(x_0, x_i) := a_i(-x_0, x_i)$. There is also an action of X_0 on X_{i+j} given by

$$\begin{aligned} a_{i+j} : X_0 \times X_{i+j} &\longrightarrow X_{i+j} \\ a_{i+j}(x_0, (x_i, x_j)) &\mapsto (a_i(x_0, x_i), x_j) \end{aligned}$$

Finally, the inverse of an element \mathbf{X}_i specified by the triple

$$(X_i, f_i, a_i) \quad (\text{A.9})$$

is given by

$$\mathbf{X}_{-i} = (X_i, f_i, a_i^{-1}). \quad (\text{A.10})$$

We end by showing that this is really the inverse, i.e.

$$\mathbf{X}_i + \mathbf{X}_{-i} = \mathbf{X}_0 \quad (\text{A.11})$$

For this, we observe that the map defined by

$$\varphi : X_i \times_B X_i \longrightarrow X_0, \quad \varphi(x_i, \tilde{x}_i) = x_i - \tilde{x}_i \quad (\text{A.12})$$

becomes an isomorphism after we divide the LHS by the following identification

$$(x_i, \tilde{x}_i) \sim (y_j, \tilde{y}_j) \Leftrightarrow \exists x_0 \in X_0 \text{ s.t. } (a_i(x_0, x_i), a_i(x_0, \tilde{x}_i)) = (y_i, \tilde{y}_i). \quad (\text{A.13})$$

However this is precisely the equivalence relation on $X_i \times_B X_i^{-1}$ as given in (A.8) if one takes into account that a_i acts as its inverse on the second factor.

Applying the above results to the case of the cubic, which has an order three element in the TS-group, we see that it consists of only two different geometries: the cubic and its Jacobian. However, we obtain three elements since the cubic can be endowed with two different actions of the Jacobian, namely a chosen action and its inverse. The two cubics obtained this way are their respective inverses in the TS-group.

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